College and Career Readiness Standards for Mathematics

Draft for Review and Comment

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Contents ₁

Introduction

The *College and Career Readiness Standards for Mathematics* consist of three interconnected parts: a Standard for Mathematical Practice, ten Standards for Mathematical Content, and a set of Example Tasks.

The Standard for Mathematical Practice has six Core Practices that describe the way proficient students approach mathematics. Proficient students attend to precision, construct viable arguments, make sense of complex problems and persevere in solving them, look for hidden structure, note regularity in repeated reasoning, and use technology intelligently. This approach to mathematics is an essential part of being ready for college and career.

The Standards for Mathematical Content form the backbone of this document. Each of these ten standards consists of Core Concepts, Core Skills, and a description of the student's Coherent Understanding. Students who encounter the subject with a focus on coherence will be better able to learn more mathematics at a deeper level and be better able to access and apply the mathematics they know. The ten Standards for Mathematical Content pull together topics previously studied and look ahead toward topics in further coursework and training programs.

The Standards for Mathematical Content are designed to draw greater attention to powerful organizing principles in mathematics, such as functional relationships or the laws of arithmetic. They also allow important distinctions to be made more clearly, such as that between Expressions and Equations. And they surface the deep connections that often underlie mathematical coherence, such as the blending of algebra with geometry represented by Coordinates. These ten are not categories or buckets of topics to cover; they are standards. They describe the coherence students need and deserve as they go forward to their mathematical futures.

The third component of the *College and Career Readiness Standards for Mathematics* is a Web-based collection of Example Tasks that exemplifies the variety of performances required. High standards demand that students *use* their knowledge, skills and good practices to solve problems from a variety of contexts, both within mathematics and from the world outside. Example Tasks exemplify the range and variety of use that is expected. Teachers and designers of curriculum and assessment will find in the collection of examples a guide to what these standards mean. Over time, the collection of tasks will grow.

Together, these three components establish an evidence-based standard for college and career readiness. The *College and Career Readiness Standards for Mathematics* have been created with attention to the expectations of the highest achieving countries. They have focus and depth, emphasizing the understanding of and connections among topics that are most important for success regardless of a student's pathway after reaching these standards.

A primary goal of developing these standards is to enable students to achieve *mathematical proficiency* (see sidebar). Students are expected to understand the knowledge described in the Core Concepts and in the Coherent Understandings at a depth that enables them to reason with that knowledge—to analyze, interpret and evaluate mathematical problems, make deductions, and justify results. The Core Skills are meant to be used strategically and adaptively to solve problems. Students' knowledge and skills come to life and take their value when melded with the ways they approach mathematics—as described by the Core Practices.

The specific verbs used to describe concepts and skills in these standards are not meant to limit or indicate levels of any taxonomy. Although using verbs to indicate levels of depth has been a common practice in this country's standards writing, high performing nations do not use verbs in this way. They describe depth and practices first in separate sections of their syllabi. We have adopted the high performing countries' practice of focusing on a clear statement of what mathematics should be learned when writing standards for knowledge and skills.

Instruction, curriculum and assessment designed to achieve these standards should range over all strands of proficiency in *Adding It Up*, all depths of knowledge in Norman L. Webb's Depth of Knowledge taxonomy, all levels of Bloom's Taxonomy, and all levels of cognitive demand. In the Core Skills and Core Practices we have sometimes used terms like "explore" to indicate a lighter treatment with a goal of awareness and experience rather than proficiency. We have used Example Tasks to show the depth of knowledge and deployment of skills expected.

From *Adding it up: Helping children learn mathematics* (National Research Council, 2001, p. 116):

Recognizing that no term captures completely all aspects of expertise, competence, knowledge, and facility in mathematics, we have chosen mathematical proficiency to capture what we believe is necessary for anyone to learn mathematics successfully. Mathematical proficiency, as we see it, has five components, or strands:

conceptual understanding—comprehension of mathematical concepts, operations, and relations

procedural fluency—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately

strategic competence—ability to formulate, represent, and solve mathematical problems

adaptive reasoning—capacity for logical thought, reflection, explanation, and justification

productive disposition—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

These standards are measurable; that is, they are observable and verifiable through the broad spectrum of student performances that may be assessed during classroom observation, school-based examinations and large-scale testing. The *College and Career Readiness Standards for Mathematics* can guide the development of assessment frameworks that distribute the assessment responsibilities across multiple levels of the educational system: state, district, school and teacher.

Students reaching these levels will be prepared for non-remedial college mathematics courses and will be prepared for training programs for career-level jobs; however, the *College and Career Readiness Standards for Mathematics* should not be construed as grade twelve exit standards. Students interested in STEM fields, and those who wish to go beyond for other reasons, will need to reach these standards before their senior year in order to have time to include additional mathematics. A number of pathways for advanced learning are possible and may be integrated throughout the high school experience and beyond.

The Common Core State Standards Initiative

The *College and Career Readiness Standards for Mathematics* will anchor the next phase of the Common Core State Standards Initiative: development of K–12 Mathematics Standards. Those K–12 Standards are in turn expected to guide the development of a next generation of assessments, developed collaboratively by multiple states. The K–12 Mathematics Standards will serve as a guide and tool for aligning instruction, curriculum, assessment, teacher supports, and systems of accountability. To ensure alignment, the Standard for Mathematical Practice, the Standards for Mathematical Content, and the Example Tasks should all be taken into account.

Overview of the Mathematical Practice Standard

Attend to precision.
Construct viable arguments.
Make sense of complex problems and persevere in solving them.
Look for structure.
Look for and express regularity in repeated reasoning.
Make strategic decisions about the use of technological tools.

Overview of the Mathematical Content Standards

Number. Procedural fluency in operations with real numbers and strategic competence in approximation are grounded in an understanding of place value. The rules of arithmetic govern operations on numbers and extend to operations in algebra.

Quantity. A quantity is an attribute of an object or phenomenon that can be specified using a number and a unit, such as 2.7 centimeters, 42 questions or 28 miles per gallon.

Expressions. Expressions use numbers, variables and operations to describe computations. The rules of arithmetic, the use of parentheses and the conventions about order of operations assure that the computation has a well-determined value.

Equations. An equation is a statement that two expressions are equal. Solutions to an equation are the values of the variables in it that make it true.

Functions. Functions model situations where one quantity determines another. Because nature and society are full of dependencies, functions are important tools in the construction of mathematical models.

Modeling. Modeling uses mathematics to help us make sense of the real world—to understand quantitative relationships, make predictions, and propose solutions.

Shape. From only a few axioms, the deductive method of Euclid generates a rich body of theorems about geometric objects, their attributes and relationships.

Coordinates. Applying a coordinate system to Euclidean space connects algebra and geometry, resulting in powerful methods of analysis and problem solving.

Probability. Probability assesses the likelihood of an event in a situation that involves randomness. It quantifies the degree of certainty that an event will happen as a number from 0 through 1.

Statistics. Decisions or predictions are often based on data—numbers in context. These decisions or predictions would be easy if the data always sent a clear message, but the message is often obscured by variability in the data.

How Evidence Informed Decisions in Drafting the Standards

The Common Core State Standards Initiative builds on a generation of standards efforts led by states and national organizations. On behalf of the states, we have taken a step toward the next generation of standards that are aligned to college- and career-ready expectations and are internationally benchmarked. These standards are grounded in evidence from many sources that shows that the next generation of standards in mathematics must be focused on deeper, more thorough understanding of more fundamental mathematical ideas and higher mastery of these fewer, more useful skills.

The evidence that supports this new direction comes from a variety of sources. International comparisons show that high performing countries focus on fewer topics and that the U.S. curriculum is "a mile wide and an inch deep." Surveys of college faculty show the need to shift away from high school courses that merely survey advanced topics, toward courses that concentrate on developing an understanding and mastery of ideas and skills that are at the core of advanced mathematics. Reviews of data on student performance show the large majority of U.S. students are not mastering the mile wide list of topics that teachers cover.

The evidence tells us that in high performing countries like Singapore, the gap between what is taught and what is learned is relatively smaller than in Malaysia or the U.S. states. Malaysia's standards are higher than Singapore's, but their performance is much lower. One could interpret the narrower gap in Singapore as evidence that they actually use their standards to manage instruction; that is, Singapore's standards were set within the reach of hard work for their system and their population. Singapore's Ministry of Education flags its webpage with the motto, "Teach Less, Learn More." We accepted the challenge of writing standards that could work that way for U.S. teachers and students: By providing focus and coherence, we could enable more learning to take place at all levels.

However, a set of standards cannot be simplistically "derived" from any body of evidence. It is more accurate to say that we used evidence to inform our decisions. A few examples will illustrate how this was done.

For example, systems of linear equations are covered by all states, yet students perform surprisingly poorly on this topic when assessed by ACT. We determined that systems of linear equations have high coherence value, mathematically; that this topic is included by all high performing nations; and that it has moderately high value to college faculty. Result: We included it in our standards.

A different and more complex pattern of evidence appeared with families of functions. Again we found that students performed poorly on problems related to many advanced functions (trigonometric, logarithmic, quadratic, exponential, and so on). Again we found that a number of states cover them, even though college faculty rated them lower in value. High performing countries include this material, but with different degrees of demand. We decided that we had to carve a careful line through these topics so that limited teaching resources could focus where most important. We decided that students should

develop deep understanding and mastery of linear and exponential functions. They should also have familiarity with other families of functions, and apply their algebraic, modeling and problem solving skills to them—but not develop in-depth technical mastery and understanding. Thus we defined two distinct levels of attention and identified which families of functions got which level of attention.

Why were exponential functions selected for intensive focus in the Functions standard instead of, say, quadratic functions? What tipped the balance was the high coherence value of exponential functions in supporting modeling and their wide utility in work and life. Quadratic functions were also judged to be well supported by expectations defined under Expressions and Equations.

These examples indicate the kind of reasoning, informed by evidence, that it takes to design standards aligned to the demands of college and career readiness in a global economy. We considered inclusion in international standards, requirements of college and the workplace, surveys of college faculty and the business community, and other sources of evidence. As we navigated these sometimes conflicting signals, we always remained aware of the finiteness of instructional resources and the need for deep mathematical coherence in the standards.

At the end of this document, there is a listing of a number of sources that played a role in the deliberations described above and more generally throughout the process to inform our decisions. A hyperlinked version of the bibliography can be found online at www.corestandards.org.

College and Career Readiness Standards for Mathematics

Mathematical Practice

Proficient students expect mathematics to make sense. They take an active stance in solving mathematical problems. When faced with a non-routine problem, they have the courage to plunge in and try something, and they have the procedural and conceptual tools to carry through. They are experimenters and inventors, and can adapt known strategies to new problems. They think strategically.

Students who engage in these practices discover ideas and gain insights that spur them to pursue mathematics beyond the classroom walls. They learn that effort counts in mathematical achievement.[a] These are practices that expert mathematical thinkers encourage in apprentices. Encouraging these practices in our students should be as much a goal of the mathematics curriculum as is teaching specific content topics and procedures.[b] Taken together with the Standards for Mathematical Content, they support productive entry into college courses or career pathways.

Core Practices · Students can and do:

1 Attend to precision.

Mathematically proficient students organize their own ideas in a way that can be communicated precisely to others, and they analyze and evaluate others' mathematical thinking and strategies noting the assumptions made. They clarify definitions. They state the meaning of the symbols they choose, are careful about specifying units of measure and labeling axes, and express their answers with an appropriate degree of precision. Rather than saying, "let v be speed and let t be time," they would say "let v be the speed in meters per second and let t be the elapsed time in seconds from a given starting time." They recognize that when someone says the population of the United States in June 2008 was 304,059,724, the last few digits indicate unwarranted precision.

2 Construct viable arguments.

Mathematically proficient students understand and use stated assumptions, definitions and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They break things down into cases and can recognize and use counterexamples. They use logic to justify their conclusions, communicate them to others and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose.

3 Make sense of complex problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They consider analogous problems, try special cases and work on simpler forms. They evaluate their progress and change course if necessary. They try putting algebraic expressions into different forms or try changing the viewing window on their calculator to get the information they need. They look for correspondences between equations, verbal descriptions, tables, and graphs. They draw diagrams of relationships, graph data, search for regularity and trends, and construct mathematical models. They check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?"

4 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern. For example, in $x^2 + 5x + 6$ they can see the 5 as 2 + 3 and the 6 as 2 × 3. They recognize the significance of an existing line in a geometric figure and can add an auxiliary line to make the solution of a problem clear. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects. For example, by seeing $5 - 3(x - y)^2$ as 5 minus a positive number times a square, they see that it cannot be more than 5 for any real numbers x and y.[b]

5 Look for and express regularity in repeated reasoning.

Mathematically proficient students pay attention to repeated calculations as they carry them out, and look both for general algorithms and for shortcuts. For example, by paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, they might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel in the expansions of $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ leads to the general formula for the sum of a geometric series. As they work through the solution to a problem, proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.[b]

6 Make strategic decisions about the use of technological tools.

Mathematically proficient students consider the available tools when solving a mathematical problem, whether pencil and paper, ruler, protractor, graphing calculator, spreadsheet, computer algebra system, statistical package, or dynamic geometry software. They are familiar enough with all of these tools to make sound decisions about when each might be helpful. They use mathematical understanding and estimation strategically, attending to levels of precision, to ensure appropriate levels of approximation and to detect possible errors. They are able to use these tools to explore and deepen their understanding of concepts.

(a) For the importance of students' beliefs about effort, see the National Mathematics Advisory Panel's Report of the Task Group on Learning Processes, p. 4-10 (2008). (b) Cuoco, A., Goldenberg, E. P., and Mark, J., *Journal of Mathematical Behavior*, 15 (4), 375-402, 1996; *Focus in High School Mathematics*. Reston, VA: NCTM, in press; Harel, G., What is Mathematics? A Pedagogical Answer to a Philosophical Question. In R. B. Gold & R. Simons (Eds.), *Current Issues in the Philosophy of Mathematics From the Perspective of Mathematicians*, Mathematical Association of America, 2008.

Number

Core Concepts · Students understand that:

A The real numbers include the rational numbers and are in one-to-one correspondence with the points on the number line.

B Quantities can be compared using division, yielding rates and ratios.

C A fraction can represent the result of dividing the numerator by the denominator; equivalent fractions have the same value.

D Place value and the rules of arithmetic form the foundation for efficient algorithms.

A Coherent Understanding of Number. Procedural fluency in operations with real numbers and strategic competence in approximation are grounded in an understanding of place value. The rules of arithmetic govern operations on numbers and extend to operations in algebra:

- Numbers can be added in any order with any grouping and multiplied in any order with any grouping.
- Adding 0 and multiplying by 1 both leave a number unchanged.
- All numbers have additive inverses, and all numbers except zero have multiplicative inverses.
- Multiplication distributes over addition.

Subtraction and division are defined in terms of addition and multiplication, so are also governed by these rules.

The place value system bundles units into 10s, then 10s into 100s, and so on, providing an efficient way to name large numbers. Subdividing in a similar way extends this to the decimal system, which provides an address system for locating all real numbers on the number line with arbitrarily high accuracy. Place value is the basis for efficient algorithms, reducing much computation to single-digit arithmetic. Mental computation strategies also make opportunistic use of the rules of arithmetic, as when the product 5×177×2 is computed at a glance to obtain 1770, rather than methodically working from left to right.

An estimate may be more appropriate than an exact value, for example, when you want to know the number of calories in a meal. Often a result is reported using fewer digits than were calculated. A mature number sense includes having rules of thumb about how much accuracy is appropriate and understanding that accuracy to more than a few decimal places often takes substantial effort. Estimation and approximation are also useful in checking calculations.

Rational numbers represented as fractions can be located on the number line by seeing them as numbers expressed in different units; for example, 3/5 is 3 units, where each unit is 1/5. However, rational numbers do not fill out the number line. There are also irrational numbers, such as π or $\sqrt{2}$. Each point on the number line then corresponds to a real number that is either rational or irrational.

Connections to Expressions, Functions and Coordinates. The rules of arithmetic govern the manipulations of expressions and functions. Two perpendicular number lines define the coordinate plane.

Core Skills · Students can and do:

1 Compare numbers and make sense of their magnitude.

> Include positive and negative numbers expressed as fractions, decimals, powers, and roots. Limit to square and cube roots. Include very large and very small numbers and the use of scientific notation.

2 Know when and how to use standard algorithms, and perform them flexibly, accurately and efficiently.*

3 Use mental strategies and technology to formulate, represent and solve problems.**

4 Solve multi-step problems involving fractions and percentages.

> Include situations such as simple interest, tax, markups/markdowns, gratuities and commissions, fees, percent increase or decrease, percent error, expressing rent as a percentage of take-home pay, and so on.

5 Use estimation and approximation to solve problems.

> Include evaluating answers for their reasonableness, detecting errors, and giving answers to an appropriate level of precision.

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[*] This aligns with the concept of procedural fluency as in the National Research Council report *Adding it up: Helping children learn mathematics*. Specifically, "*Procedural fluency* refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently" (p. 121).

[**] This aligns with the concept of strategic competence as described in *Adding it up*. "*Strategic competence* refers to the ability to formulate mathematical problems, represent them, and solve them" (p. 124).

Quantity

Core Concepts · Students understand that:

A The value of a quantity is not specified unless the units are named or understood from the context.

B Quantities can be added and subtracted only when they are of the same type (length, area, speed, etc.).

C Quantities can be multiplied or divided to create new types of quantities, called derived quantities.

A Coherent Understanding of Quantity. A quantity is an attribute of an object or phenomenon that can be specified using a number and a unit, such as 2.7 centimeters, 42 questions or 28 miles per gallon.

The length of a football field and the speed of light are both quantities. If we choose units of miles per second, then the speed of light has a value of approximately 186,000 miles per second. But the speed of light need not be expressed in miles per second; it may be approximated by 3×10^8 meters per second or in any other unit of speed. Bare numerical values such as 186,000 do not describe quantities unless they are paired with units.

Speed (distance divided by time), rectangular area (length multiplied by length), density (mass divided by volume), and population density (number of people divided by land area) are examples of derived quantities, obtained by multiplying or dividing quantities.

It can make sense to add two quantities, such as when a child 51 inches tall grows 3 inches to become 54 inches tall. To be added or subtracted, quantities must be of the same type (length, area, speed, etc.); to add or subtract their values, the quantities must be expressed in the same units. Converting quantities to have the same units is like converting fractions to have a common denominator before adding or subtracting. But, even when quantities have the same units it does not always make sense to add them. For example, if a wooded park with 300 trees per acre is next to a field with 30 trees per acre, they do not have 330 trees per acre.

Doing algebra with units in a calculation reveals the units of the answer, and can help reveal a mistake if, for example, the answer comes out to be a distance when it should be a speed.

Connections to Number, Expressions, Equations, Functions, Modeling and Statistics. Operations described under Number and Expressions govern the operations one performs on quantities, including the units involved. Quantity is an integral part of any application of mathematics, and has connections to solving problems using data, equations, functions and modeling.

Core Skills · Students can and do:

1 Know when and how to convert units in computations.

> Include the addition and subtraction of quantities of the same type expressed in different units; averaging data given in mixed units; converting units for derived quantities such as density and speed.

2 Use and interpret quantities and units correctly in algebraic formulas.

> Include specifying units when defining variables and attending to units when writing expressions and equations.

3 Use and interpret quantities and units correctly in graphs and data displays.

> Include function graphs, data tables, scatterplots and other visual displays of dimensioned data.

4 Use units as a way to understand problems and to guide the solution of multi-step problems.

> Include examples such as acceleration; currency conversions; people-hours; social science measures, such as deaths per 100,000; and general rates, such as points per game.

Expressions

Core Concepts · Students understand that:

A Expressions are constructions built up from numbers, variables, and operations, which have a numerical value when each variable is replaced with a number.

B Complex expressions are made up of simpler expressions.

C The rules of arithmetic can be applied to transform an expression without changing its value.

D Rewriting expressions in equivalent forms serves a purpose in solving problems.

A Coherent Understanding of Expressions. Expressions use numbers, variables and operations to describe computations. The rules of arithmetic, the use of parentheses and the conventions about order of operations assure that the computation has a well-determined value.

Reading an expression with comprehension involves analysis of its underlying structure, which may suggest a different but equivalent way of writing it that exhibits some different aspect of its meaning. For example, $p + 0.05p$ can be interpreted as the addition of a 5% tax to a price p. But rewriting $p + 0.05p$ as $1.05p$ shows that adding a tax is the same as multiplying by a constant factor.

Algebraic manipulations are based on the conventions of algebraic notation and the rules of arithmetic. Heuristic mnemonic devices are not a substitute for procedural fluency. For example, factoring, expanding, collecting like terms, the rules for interpreting minus signs next to parenthetical sums, and adding fractions with a common denominator are all instances of the distributive law; the definitions for negative and rational exponents are based on the extension of the exponent laws for positive integers. The laws of exponents connect multiplication of numbers to addition of exponents and thus express the deep relationship between addition and multiplication captured by the parallel nature of the rules of arithmetic for these operations.

Complex expressions are made up of simpler expressions using arithmetic operations and substitution. When simple expressions within more complex expressions are treated as single quantities, or chunks, the underlying structure of the larger expression may be more evident.

Connections to Equations and Functions. Setting expressions equal to each other leads to equations. Expressions can define functions of the variables that appear in them, with equivalent expressions defining the same function.

Core Skills · Students can and do:

1 See structure in expressions.

For example, recognize: that the expressions $x^4 - y^4$ and $(x + y)^2 - (x - y)^2$ are differences of squares; that there are different ways to rewrite the latter expression, e.g., by expanding and collecting like terms or by factoring as a difference of squares; that p is a common factor in $p + 0.025p$; that an expression in the form $(x - 3)^2 + 14$ reveals its minimum value.

2 Manipulate simple expressions.

Show procedural fluency in the following cases: factoring out common terms; factoring expressions with quadratic structure; writing in standard form sums, differences, and products of polynomials. Include completing the square and rewriting in standard form sums, differences, products, and quotients of simple rational expressions; rewriting expressions with negative exponents and those involving square or cube roots of a single term involving exponents.

3 Define variables and write an expression to represent a quantity in a problem.

Include contextual problems.

4 Interpret an expression that represents a quantity in terms of the context.

Include interpreting parts of an expression, such as terms, factors and coefficients.

Equations

Core Concepts · Students understand that:

A An equation is a statement that two expressions are equal.

B The solutions of an equation are the values of the variables that make the resulting numerical statement true.

C The steps in solving an equation are guided by understanding and justified by logical reasoning.

D Equations not solvable in one number system may have solutions in a larger number system.

A Coherent Understanding of Equations. An equation is a statement that two expressions are equal. Solutions to an equation are the values of the variables in it that make it true. If the equation is true for all values of the variables, then we call it an identity; identities are often discovered by manipulating one expression into another.

The solutions of an equation in one variable form a set of numbers; the solutions of an equation in two variables form a set of ordered pairs, which can be graphed in the plane. Equations can be combined into systems to be solved simultaneously.

An equation can be solved by successively transforming it into one or more simpler equations. The process is governed by deductions based on the properties of equality. For example, one can add the same constant to both sides without changing the solutions, but squaring both sides might lead to extraneous solutions. Strategic competence in solving includes looking ahead for productive manipulations and anticipating the nature and number of solutions.

Some equations have no solutions in a given number system, stimulating the formation of expanded number systems (integers, rational numbers, real numbers and complex numbers).

A formula is a type of equation. The same solution techniques used to solve equations can be used to rearrange formulas. For example, the formula for the area of a trapezoid, $A = \left(\frac{b_1 + b_2}{2}\right)h$, can be solved for h using the same deductive process.

Inequalities can be solved in much the same way as equations. Many, but not all, of the properties of equality extend to the solution of inequalities.

Connections to Functions, Coordinates, and Modeling. Equations in two variables may define functions. Asking when two functions have the same value leads to an equation; graphing the two functions allows for the approximate solution of the equation. Equations of lines involve coordinates, and converting verbal descriptions to equations is an essential skill in modeling.

Core Skills · Students can and do:

1 Understand a problem and formulate an equation to solve it.

> Extend to inequalities and systems.

2 Solve equations in one variable using manipulations guided by the rules of arithmetic and the properties of equality.

> Solve linear equations with procedural fluency. For quadratic equations, include solution by inspection, by factoring, or by using the quadratic formula. Understand that the quadratic formula comes from completing the square. Include simple absolute value equations solvable by direct inspection and by understanding the interpretation of absolute value as distance.

3 Rearrange formulas to isolate a quantity of interest.

> Exclude cases that require extraction of roots or inverse functions.

4 Solve systems of equations.

> Focus on pairs of simultaneous linear equations in two variables. Include algebraic techniques, graphical techniques and solving by inspection.

5 Solve linear inequalities in one variable and graph the solution set on a number line.

> Emphasize solving the associated equality and determining on which side of the solution of the associated equation the solutions to the inequality lie.

6 Graph the solution set of a linear inequality in two variables on the coordinate plane.

> Emphasize graphing the associated equation, using a dashed or solid line as appropriate and shading to indicate the half-plane on which the solutions to the inequality lie.

Functions

Core Concepts · Students understand that:

A A function is a rule, often defined by an expression, that assigns a unique output for every input.

B The graph of a function f is a set of ordered pairs $(x, f(x))$ in the coordinate plane.

C Functions model situations where one quantity determines another.

D Common functions occur in families where each member describes a similar type of dependence.

A Coherent Understanding of Functions. Functions model situations where one quantity determines another. For example, the return on \$10,000 invested at an annualized percentage rate of 4.25% is a function of the length of time the money is invested. Because nature and society are full of dependencies between quantities, functions are important tools in the construction of mathematical models.

In school mathematics, functions usually have numerical inputs and outputs and are often defined by an algebraic expression. For example, the time in hours it takes for a plane to fly 1000 miles is a function of the plane's average ground speed in miles per hour, v; the rule $T(v) = 1000/v$ expresses this relationship algebraically and defines a function whose name is T.

The set of possible inputs to a function is called its domain. We often infer the domain to be all inputs for which the expression defining a function has a value, or for which the function makes sense in a given context. The graph of a function is a useful way of visualizing the relationship the function models, and manipulating the expression for a function can throw light on the function's properties.

Two important families of functions characterized by laws of growth are linear functions, which grow at a constant rate, and exponential functions, which grow at a constant percent rate. Linear functions with an initial value of zero describe proportional relationships.

Connections to Expressions, Equations, Modeling and Coordinates. Determining an output value for a particular input involves evaluating an expression; finding inputs that yield a given output involves solving an equation. The graph of a function f is the same as the solution set of the equation $y = f(x)$. Questions about when two functions have the same value lead to equations, whose solutions can be visualized from the intersection of the graphs. Since functions describe relationships between quantities, they are frequently used in modeling. Sometimes functions are defined by a recursive process, which can be modeled effectively using a spreadsheet or other technology.

Core Skills · Students can and do:

1 Recognize proportional relationships and solve problems involving rates and ratios.

> Include being able to express proportional relationships as functions.

2 Describe the qualitative behavior of common types of functions using graphs and tables.

> Identify: intercepts; intervals where the function is increasing, decreasing, positive or negative; relative maximums and minimums; symmetries; end behavior; and periodicity. Use technology to explore the effects of parameter changes on the graphs of linear, power, quadratic, polynomial, simple rational, exponential, logarithmic, sine and cosine, absolute value and step functions.

3 Analyze functions using symbolic manipulation.

> Include slope-intercept and point-slope form of linear functions; vertex form of quadratic functions to identify symmetry and find maximums and minimums; factored form to find zeros. Use manipulations as described under Expressions.

4 Use the families of linear and exponential functions to solve problems.

> For linear functions $f(x) = mx + b$, understand b as the intercept or initial value and m as the slope or rate of change. For exponential functions $f(x) = a \cdot b^x$, understand a as the intercept or initial value and b as the growth factor.

5 Find and interpret rates of change.

> Compute the rate of change of linear functions and make qualitative observations about how the rate of change varies for nonlinear functions.

Modeling

Core Concepts · Students understand that:

A Mathematical models involve choices and assumptions that abstract key features from situations to help us solve problems.

B Even very simple models can be useful.

A Coherent Understanding of Modeling. Modeling uses mathematics to help us make sense of the real world—to understand quantitative relationships, make predictions, and propose solutions.

A model can be very simple, such as a geometric shape to describe a physical object like a coin. Even so simple a model involves making choices. It is up to us whether to model the solid nature of the coin with a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. For some purposes, we might even choose to adjust the right circular cylinder to model more closely the way the coin deviates from the cylinder.

In any given situation, the model we devise depends on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models we can create and analyze is constrained as well by the limitations of our mathematical and technical skills. For example, modeling a physical object, a delivery route, a production schedule, or a comparison of loan amortizations each requires different sets of tools. Networks, spreadsheets and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations. One of the insights provided by mathematical modeling is that essentially the same mathematical structure might model seemingly different situations.

The basic modeling cycle is one of (1) identifying the key features of a situation, (2) creating geometric, algebraic or statistical objects that describe key features of the situation, (3) analyzing and performing operations on these objects to draw conclusions and (4) interpreting the results of the mathematics in terms of the original situation. Choices and assumptions are present throughout this cycle.

Connections to Quantity, Equations, Functions, Shape, Coordinates and Statistics. Modeling makes use of shape, data, graphs, equations and functions to represent real-world quantities and situations.

Core Skills · Students can and do:

1 Model numerical situations.

Include readily applying the four basic operations in combination to solve multi-step quantitative problems with dimensioned quantities; making estimates to introduce numbers into a situation and get problems started; recognizing proportional or near-proportional relationships and analyzing them using characteristic rates and ratios.

2 Model physical objects with geometric shapes.

Include common objects that can reasonably be idealized as two- and three-dimensional geometric shapes. Identify the ways in which the actual shape varies from the idealized geometric model.

3 Model situations with equations and inequalities.

Include situations well described by a linear inequality in two variables or a system of linear inequalities defining a region in the plane.

4 Model situations with common functions.

Include situations well described by linear, quadratic or exponential functions; and situations that can be well described by inverse variation ($f(x) = k/x$). Include identifying a family of functions that models features of a problem, and identifying a particular function of that family and adjusting it to fit by changing parameters. Understand the recursive nature of situations modeled by linear and exponential functions.

5 Model situations using probability and statistics.

Include using simulations to model probabilistic situations; describing the shape of a distribution of values and summarizing a distribution with measures of center and variability; modeling a bivariate relationship using a trend line or a regression line.

6 Interpret the results of applying a model and compare models for a particular situation.

Include realizing that models seldom fit exactly and so there can be error; identifying simple sources of error and being careful not to over-interpret models. Include recognizing that there can be many models that relate to a situation, that they can capture different aspects of the situation, that they can be simpler or more complex, and that they can have a better or worse fit to the situation and the questions being asked.

Shape

Core Concepts · Students understand that:

A Shapes and their parts, attributes, and measurements can be analyzed deductively.*

B Congruence, similarity, and symmetry can be analyzed using transformations.

C Mathematical shapes model the physical world, resulting in practical applications of geometry.

D Right triangles and the Pythagorean theorem are central to geometry and its applications, including trigonometry.

A Coherent Understanding of Shape. From only a few axioms, the deductive method of Euclid generates a rich body of theorems about geometric objects, their attributes and relationships. Once understood, those attributes and relationships can be applied in diverse practical situations—interpreting a schematic drawing, estimating the amount of wood needed to frame a sloping roof, rendering computer graphics, or designing a sewing pattern for the most efficient use of material.

Understanding the attributes of geometric objects often relies on measurement: a circle is a set of points in a plane at a fixed distance from a point; a cube is bounded by six squares of equal area; when two parallel lines are crossed by a transversal, pairs of corresponding angles are congruent.

The concepts of congruence, similarity and symmetry can be united under the concept of geometric transformation. Reflections and rotations each explain a particular type of symmetry, and the symmetries of an object offer insight into its attributes—as when the reflective symmetry of an isosceles triangle assures that its base angles are congruent. Applying a scale transformation to a geometric figure yields a similar figure. The transformation preserves angle measure, and lengths are related by a constant of proportionality. If the constant of proportionality is one, distances are also preserved (so the transformation is a rigid transformation) and the figures are congruent.

The definitions of sine, cosine and tangent for acute angles are founded on right triangle similarity, and, with the Pythagorean theorem, are fundamental in many practical and theoretical situations.

Connections to Coordinates, Functions and Modeling. The Pythagorean theorem is a key link between geometry, measurement and distance in the coordinate plane. Parameter changes in families of functions can be interpreted as transformations applied to their graphs and those functions, as well as geometric objects in their own right, can be used to model contextual situations.

*In this document, deductive analysis aligns with the notion of adaptive reasoning as defined in *Adding it Up*, and includes empirical exploration, informal justification, and formal proof.

Core Skills · Students can and do:

1 Use multiple geometric properties to solve problems involving geometric figures.

> Properties include: measures of interior angles of a triangle sum to 180°; vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; measures of supplementary angles sum to 180°; two lines parallel to a third are parallel to each other; points on a perpendicular bisector of a segment are exactly those equidistant from the segment's endpoints; and a line tangent to a circle is perpendicular to the radius meeting it.

2 Prove theorems, test conjectures and identify logical errors.

> Include theorems establishing the properties in Core Skill 1 and other theorems about angles, parallel and perpendicular lines, similarity and congruence of triangles.

3 Construct and interpret representations of geometric objects.

> Include classical construction techniques and construction techniques supported by modern technologies. Include moving between two-dimensional representations and the three-dimensional objects they represent, such as in schematics, assembly instructions, perspective drawings and multiple views.

4 Solve problems involving measurements.

> Include measurement (length, angle measure, area, surface area, and volume) of a variety of figures and shapes in two- and three-dimensions. Compute measurements using formulas and by decomposing complex shapes into simpler ones.

5 Solve problems involving similar triangles and scale drawings.

> Include computing actual lengths, areas and volumes from a scale drawing and reproducing a scale drawing at a different scale.

6 Apply properties of right triangles and right triangle trigonometry to solve problems.

> Include using the Pythagorean theorem and properties of special right triangles, and applying sine, cosine and tangent to determine lengths and angle measures of right triangles. Use right triangles and their properties to solve real-world problems. Limit angle measures to degrees.

Coordinates

Core Concepts · Students understand that:

A Locations in the plane or in space can be specified by pairs or triples of numbers called coordinates.

B Coordinates link algebra with geometry and allow methods in one domain to solve problems in the other.

C The set of solutions to an equation in two variables forms a curve in the coordinate plane—such as a line, parabola, circle—and the solutions to systems of equations correspond to intersections of these curves.

A Coherent Understanding of Coordinates. Applying a coordinate system to Euclidean space connects algebra and geometry, resulting in powerful methods of analysis and problem solving.

Just as the number line associates numbers with locations in one dimension, a pair of perpendicular axes associates pairs of numbers with locations in two dimensions. This correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling and proof.

Coordinate geometry is a rich field for exploration. How does a geometric transformation such as a translation or reflection affect the coordinates of points? How is the geometric definition of a circle reflected in its equation?

Adding a third perpendicular axis associates three numbers with locations in three dimensions and extends the use of algebraic techniques to problems involving the three-dimensional world we live in.

Connections to Shape, Quantity, Equations and Functions. Coordinates can be used to reason about shapes. In applications, coordinate values often have units (such as meters and bushels). A one-variable equation of the form $f(x) = g(x)$ may be solved in the coordinate plane by finding intersections of the curves $y = f(x)$ and $y = g(x)$.

Core Skills · Students can and do:

1 Translate fluently between lines in the coordinate plane and their equations.

> Include predicting visual features of lines by inspection of their equations, determining the equation of the line through two given points, and determining the equation of the line with a given slope passing through a given point.

2 Identify the correspondence between parameters in common families of equations and the location and appearance of their graphs.

> Include common families of equations—the graphs of $Ax + By = C$, $y = mx + b$ and $x = a$ are straight lines; the graphs of $y = a(x - h)^2 + k$ and $y = Ax^2 + Bx + C$ are parabolas; and the graph of $(x - h)^2 + (y - k)^2 = r^2$ is a circle.

3 Use coordinates to solve geometric problems.

> Include proving simple theorems algebraically, using coordinates to compute perimeters and areas for triangles and rectangles, finding midpoints of line segments, finding distances between pairs of points and determining when two lines are parallel or perpendicular.

Probability

Core Concepts · Students understand that:

A   Probability models outcomes for situations in which there is inherent randomness, quantifying the degree of uncertainty in terms of relative frequency of occurrence.

B   The law of large numbers provides the basis for estimating certain probabilities by use of empirical relative frequencies.

C   The laws of probability govern the calculation of probabilities of combined events.

D   Interpreting probabilities contextually is essential to rational decision-making in situations involving randomness.

A Coherent Understanding of Probability. Probability assesses the likelihood of an event in a situation that involves randomness. It quantifies the degree of certainty that an event will happen as a number from 0 through 1. This number is generally interpreted as the relative frequency of occurrence of the event over the long run.

   The structure of a probability model begins by listing or describing the possible outcomes for a random situation (the sample space) and assigning probabilities based on an assumption about long-run relative frequency. In situations such as flipping a coin, rolling a number cube, or drawing a card, it is reasonable to assume various outcomes are equally likely.

   Compound events constructed from these simple ones can be represented by tree diagrams and by frequency or relative frequency tables. The probabilities of compound events can be computed using these representations and by applying the additive and multiplicative laws of probability. Interpreting these probabilities relies on an understanding of independence and conditional probability, approachable through the analysis of two-way tables.

   Converting a verbally-stated problem into the symbols and relations of probability requires careful attention to words such as *and*, *or*, *if*, and *all*, and to grammatical constructions that reflect logical connections. This is especially true when applying probability models to real-world problems, where simplifying assumptions are also usually necessary in order to gain at least an approximate solution.

   Connections to Statistics and Expressions. Probability is the foundation for drawing valid conclusions from sampling or experimental data. Counting has an advanced connection with Expressions through Pascal's triangle and binomial expansions.

Core Skills · Students can and do:

1   Compute theoretical probabilities by systematically counting points in the sample space.

   Make use of symmetry and equally likely outcomes. Include permutation and combination problems as long as small numbers are involved or technology is used, so that formulas are not required.

2   Interpret probabilities of compound events using concepts of independence and conditional probability.

   Include reading conditional probabilities from two-way tables.

3   Compute probabilities of compound events.

   Make use of the additive and multiplicative laws of probability, tree diagrams and frequency or relative frequency tables in real contexts. Do not emphasize fluency with the related formulas

4   Estimate probabilities empirically.

   Include using data from simulations carried out with technology to estimate probabilities.

5   Identify and explain common misconceptions regarding probability.

   Include misconceptions about long-run versus short-run behavior of relative frequencies (the law of large numbers). Include attention to the use and misuse of probability in the media, especially in terms of interpreting charts and tables and in the contextual meaning of terms connected to probability, such as 'odds' or 'risk.'

6   Adapt probability models to solve real-world problems.

   Include the use of conditional probability to assess subsets of data (e.g., what does the data say about males and females separately). Include the use of independence as a simplifying assumption (e.g., find the probability that two students both contract the disease this year).

Statistics

Core Concepts · Students understand that:

A Statistical methods take variability into account to support making informed decisions based on quantitative studies designed to answer specific questions.

B Visual displays and summary statistics condense the information in data sets into usable knowledge.

C Randomness is the foundation for using statistics to draw conclusions when testing a claim or estimating plausible values for a population characteristic.

D The design of an experiment or sample survey is of critical importance to analyzing the data and drawing conclusions.

A Coherent Understanding of Statistics. Decisions or predictions are often based on data—numbers in context. These decisions or predictions would be easy if the data always sent a clear message, but the message is often obscured by variability in the data. Statistics provides tools for describing variability in data and for making informed decisions that take variability into account.

Data are gathered, displayed, summarized, examined and interpreted to discover patterns. Data can be summarized by a statistic measuring center, such as mean or median, and a statistic measuring spread, such as interquartile range or standard deviation. Different distributions can be compared numerically using these statistics or visually using plots. Which statistics to compare, and what the results of a comparison might mean, depend on the question to be investigated and the real-life actions to be taken.

Randomization has two important uses in drawing statistical conclusions. First, collecting data from a random sample of a population makes it possible to draw valid conclusions about the whole population, taking variability into account. Second, randomly assigning individuals to different treatments allows a fair comparison of the effectiveness of those treatments. A statistically significant outcome is one that is unlikely to be due to chance and this can be evaluated only under the condition of randomness.

In critically reviewing uses of statistics in public media and other reports, it is important to consider the study design, how the data were collected, and the analyses employed as well as the data summaries and the conclusions drawn.

Connections to Probability, Functions and Modeling. Valid conclusions about a population depend on designed simulations or other statistical studies using random sampling or assignment and rely on probability for their interpretation. Functional models may be used to approximate data. If the data are approximately linear, the relationship may be modeled with a trend line and the strength and direction of such a relationship may be expressed through a correlation coefficient. Technology facilitates the study of statistics by making it possible to simulate many possible outcomes in a short amount of time, and by generating plots, function models, trend lines and correlation coefficients.

Core Skills · Students can and do:

1 Formulate questions that can be addressed with data. Identify the relevant data, collect and organize it to respond to the question.

Include determining whether a question can best be addressed through a sample survey, randomized experiment or observational study. Include unbiased selection for a sample and randomization of assignment to treatment for an experiment.

2 Use appropriate displays and summary statistics for data.

Include univariate, bivariate, categorical and quantitative data. Include the thoughtful selection of displays and measures of center and spread to summarize data.

3 Interpret data displays and summaries critically; draw conclusions and develop recommendations.

Include paying attention to the context of the data, interpolating or extrapolating judiciously, and examining the effects of extreme values of the data on summary statistics of center and spread. Include data sets that follow a normal distribution. Include observing and interpreting linear trends in bivariate quantitative data.

4 Draw statistical conclusions involving population means or proportions using sample data.

Conclusions should be based on simulations or other informal techniques, rather than formulas.

5 Evaluate reports based on data.

Include looking for bias or flaws in the way the data were gathered or presented, as well as unwarranted conclusions, such as claims that confuse correlation with causation.

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